Optimisation of structural health monitoring system topology based on the value of information concept

Piotr Omenzetter

The Lloyd's Register Foundation Centre for Safety and Reliability Engineering, The University of Aberdeen, Aberdeen, UK

ABSTRACT: This paper formulates the basis for a theory of optimising the structural health monitoring (SHM) system topologies based on maximising the value of information an SHM system can deliver. The value of SHM information is in overall reduction of the expected risk and is calculated using the pre-posterior Bayesian analysis. This requires, inter alia, the prior probabilities of system survival/failure and likelihoods of survival/failure state indication by the monitoring system. The paper thus starts with some basic concepts from the structural reliability theory which point out to the fact that system survival/failure probabilities are functions of local member- or cross-section-level survival/failure probabilities, and that the latter can be updated using SHM data. Then, the pre-posterior Bayesian decision tree analysis used to minimise the total risk is introduced including the probabilistic modelling of the SHM system performance and considering the probabilities of state mis-indication errors. Finally, the sensing system topology optimisation problem is stated mathematically. A simple analytical example using a truss structure with strain gauge-based SHM system rounds up the paper and illustrates the concepts discussed.

1 INTRODUTION

The existing schemes for planning optimal placement of sensors for structural monitoring often consider only vibrational responses and use general information-theoretic approaches. For example, Meo and Zumpano (2005) compared six different optimal sensor placement techniques and suggested that the effective independence-driving point residue method (Kammer and Tinker 2004) provides an efficient method for optimal sensor placement to identify the vibration characteristics. However, such schemes do not account for the many other types of measurements employed in structural health monitoring (SHM), such as static responses or corrosion depth, to name just two classes or measurands. Furthermore, they are not clearly linked to quantitative structural reliability or risk assessment methods that are necessary for quantifying the value of SHM through probabilistic Bayesian approaches inherent to pre-posterior decision analysis (Straub 2014).

Following from the above observations, this paper sketches a general theory for optimal sensor placement for assessment of structural condition, or managing the risk of structural failures, based on minimizing the failure risk using SHM data against the cost of data collection, i.e. maximizing the value of information from health monitoring. The theory is based on the pre-posterior Bayesian analysis of decision trees (Raiffa and Schlaifer 1961) and has strong links to structural reliability (Melchers 1999). This approach enables the quantification, in a probabilistic sense, of the benefits of procuring and installing an SHM system before even making the decision to employ SHM.

The paper thus starts with some basic concepts from the structural reliability theory. The focus here is on the system failure probabilities as functions of local failure probabilities, and the fact that the latter can be updated using SHM data. Then, the pre-posterior Bayesian decision tree analysis used to minimise the total risk is introduced including the probabilistic modelling of the SHM system performance and considering the probabilities of system survival/failure state mis-indication errors. Finally, the sensing system topology optimisation problem is stated mathematically. A simple analytical example using a truss structure with strain gauge-based SHM system rounds up the paper and illustrates the concepts discussed.

2 UNERLYING CONCEPTS FROM STRUCTURAL RELIABILITY THEORY

The reliability of a structural system is a function of the reliability of its members (Melchers 1999). Structural damage or failure occurs locally, taking, e.g., the form of buckling of a member, formation of a plastic hinge or a crack at a given location, etc. System failure occurs when, depending on the degree of structural redundancy, simultaneous occurrence of local failures in one or more locations turns a stable structural system into a mechanism. This is schematically illustrated in Figure 1 using as an example a pinned-pinned portal frame, for which plastic hinges can form in three places: at the left hand side corner, at the beam mid-span, and at the right hand side corner, respectively. The frame has in turn three different system bending failure modes: the beam mechanism, the sway mechanism and the mixed mechanism. (Note the previous statement ignores other possible failure modes, e.g. another sway mode where the frame would fold to the left. This can be justified by the fact that that mode would be extremely unlikely given the assumed load system. However, generally the overall system failure probability will be underestimated using our approach.)

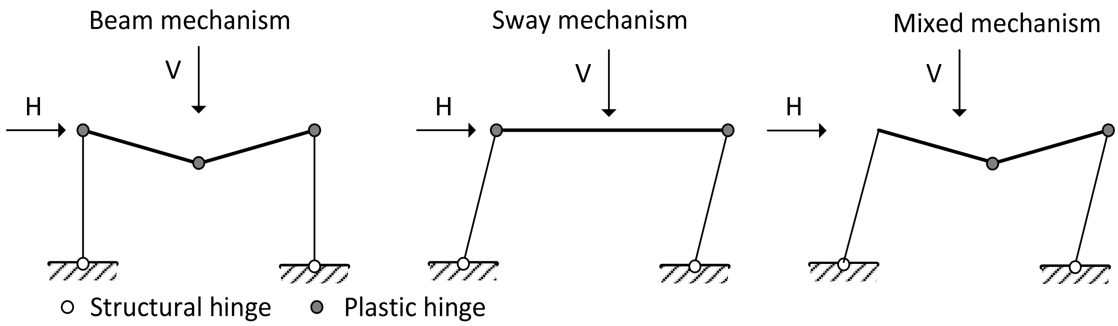

Figure 1. Local damage leading to system failure in different modes for a pinned-pinned frame.

For a general structural system, denote by $FS_j^{(i)}$ the random event that *i*-th amongst the total number of *m* members/cross-sections survives (subscript *j*=0), or, alternatively, sustains local failure (subscript *j*=1). The various, mutually exclusive, cases of simultaneous occurrence of member/cross-section survivals or local failures can be collated in the following system survival/failure state matrix:

$$\mathbf{FS}_{ij}^{(S)} = \begin{cases} 0 & \text{if in } i\text{-th system state } j\text{-th element/cross-section survives} \\ 1 & \text{if in } i\text{-th system state } j\text{-th member/cross-section fails} \end{cases} \tag{1}$$

where i=1, 2,… 2^m (i.e. the total number of possible 'binary' survival/failure states), and j=1, 2,… m. For the frame in Figure 1, the matrix would look as follows:

$$\mathbf{FS}^{(S)} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}^T \begin{matrix} \text{survival/failure (plastic hinge) at the left hand side corner} \\ \text{survival/failure (plastic hinge) at the beam mid-span} \\ \text{survival/failure (plastic hinge) at the right hand side corner} \end{matrix} \tag{2}$$

Note the matrix is presented here in its transposed form indicated by the superscript T. The 1st row of the (transposed) matrix corresponds to the left hand side corner of the frame, the 2nd row to the beam mid-span, and the 3rd row to the right hand side corner of the frame, respectively. With that system survival/failure state coding convention, the 5th column corresponds to the mixed failure mode shown in Figure 1, the 6th column to the sway mode, and the 8th column to the beam failure mode, respectively.

The probability of the occurrence of a given system survival or failure state (row of the system survival/failure state matrix, $\mathbf{FS}^{(S)}$) will be denoted as $FS_{i_1 i_2 \ldots i_m}^{(S)}$, can be expressed as the probability of the union of the corresponding local survival/failure states:

$$P\left(FS_{i_1 i_2 \ldots i_m}^{(S)}\right) = P\left(\bigcap_{j=1}^{m} FS_{i_j}^{(j)}\right), i_1, i_2, \ldots i_m = 0 \text{ or } 1 \tag{3}$$

For the use of pre-posterior analysis, outlined later in Section 3.2, the local survival/failure state probabilities are required to establish the system survival/failure state probabilities. The role of information derived from monitoring data will be to update local survival/failure state probabilities. Subsequently, these local survival/failure state probabilities will be used to update the overall system survival/failure mode probabilities. Consequences (costs) can then be assigned to failures and the various decisions and actions that can be made or undertaken using the SHM information. The overall expected cost of failure (i.e. risk) can then be calculated and compared to the cost of monitoring system. Note here that structural reliability calculations even for moderately complex systems may generally entail significant computational load.

3 FORMULASION OF SHM SYSTEM TOPOLOGY OPTIMISATION PROBLEM

3.1 Sensor topology description

Figure 2 uses a simple truss as an illustration of different monitoring system topologies that can be considered in practical applications. There are i=1, 2, … n potential 'sensor holders' at different practically relevant physical locations, and at each of them one of the j=0, 1, 2, … l sensor types can be installed (e.g. accelerometers, stain gauges, etc.); j=0 corresponds to no sensor). Note to accommodate more than one sensor or sensor type at a given physical location there may be several 'sensor holders' at any one location.) All possible sensing system topologies can be gathered in the following matrix **ST**:

$$\mathbf{ST}_{ki} = j \text{ if in topology } k \text{ in 'sensor holder' } i \text{ there is sensor of type } j;$$
$$i = 1, 2, \ldots n; \quad j = 0, 1, \ldots l; \quad k = 1, 2, \ldots (l+1)^k \tag{4}$$

where $(l+1)^k$ is the total number of technologies that can be considered. Each row of the matrix, denoted by $ST_{i_1 i_2 \ldots i_n}$, describes a particular SHM system topology.

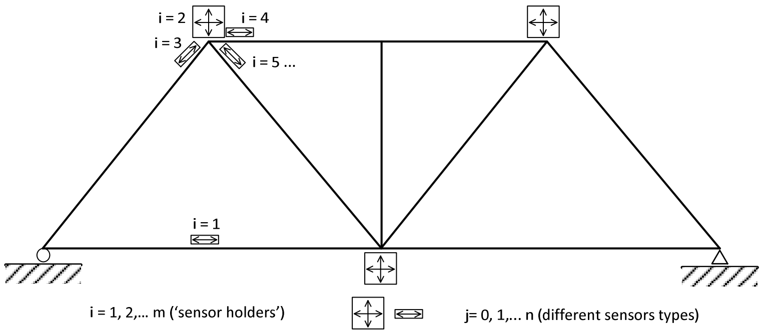

Figure 2. Example of a sensing system topology for a truss.

3.2 *Pre-posterior analysis and decision tree*

The pre-posterior analysis (Raiffa and Schlaifer 1961) and the decision tree is presented in Figure 3. Note that only branches corresponding to the cases of no monitoring and one particular topology adopted have been shown, the other topology choices having been only indicated for the sake of avoiding cluttering the figure with repetitive detail; all branches have been trimmed and indicted by '…' to avoid repetitive, but otherwise obvious, detail.

Each possible scenario in the decision tree begins with choosing a particular SHM system topology (or choosing not to adopt any SHM system) described by the corresponding row of matrix **ST** (Eq. (4)) denoted by $ST_{i_1 i_2 \ldots i_n}$. Note each choice will entail the corresponding cost of installing a particular SHM system, $C_{monit}\left(ST_{i_1 i_2 \ldots i_n}\right)$. Once the SHM system is operational, it will indicate one of the possible m survival/failure states with probability $P\left(FD^{(S)}_{i_1 i_2 \ldots i_m}\right)$. Note, the outcome from the SHM system is random in nature as in addition to correct state indications there may also be cases when the SHM system is confused.

Based on the indication of the system survival/failure state from the SHM system the operator will take an action or make a decision. If no SHM system is adopted they will have to rely on the prior survival/failure probabilities alone to make decisions. The set of possible actions/decisions is context-dependent and may include, for example, continuation of normal operations/usage/occupancy if the SHM system does not indicate damage, or closure of a bridge to traffic or evacuation of a building when it does, etc. We assume here for simplicity that the set of actions/decisions comprises only two alternatives denoted by AC_i (i=0, 1) (for a more refined case with several alternatives for decisions and failure states, see Omenzetter (2017)). The final branches of the decisions tree are the actual survival/failure states of the system, which are again random events described by their corresponding probabilities $P\left(FS^{(S)}_{i_1 i_2 \ldots i_m}\right)$ (see Section 2).

Following each of the branches of the decision tree, we can assign to each sequence of random events (SHM-indicated and actual survival/failure state occurrences) (denoted in the figure by circles) and actions/decision (denoted in the figure by squares) its likely costs or consequences,

$C_{monit}\left(ST_{00\ldots0}, AC_i, FS^{(S)}_{j_1 j_2 \cdots j_m}\right)$ or $C_{monit}\left(ST_{i_1 i_2 \ldots i_n}, FD^{(S)}_{j_1 j_2 \cdots j_m}, AC_k, FS^{(S)}_{l_1 l_2 \ldots l_m}\right)$, respectively. For example, if the SHM system indicates structural failure and a decision is made to stop operations, but there is in fact no failure (i.e. the SHM system is 'wrong' in making a false-positive damage indication), there will be undesirable consequences due to unnecessary disruption to operations (e.g. loss of service). On the other hand, if the SHM system misses an actual failure (false negative indication) this may lead to further cascading damage, casualties, etc., in addition to operational disruption. We assume here a rational decision maker, i.e. one that makes decisions that minimise the overall expected risk, and one who is neither risk-averse nor risk-seeking.

The survival/failure state probabilities $P\left(FS^{(S)}_{i_1 i_2 \ldots i_m}\right)$ introduced in Section 2 and obtainable from structural reliability considerations are prior probabilities. However, using the information from the SHM system allows updating these probabilities to their posterior values, $P\left(FS^{(S)}_{i_1 i_2 \ldots i_m} \mid FD^{(S)}_{j_1 j_2 \cdots j_m}\right)$, using the Bayes' formula:

$$P\left(FS^{(S)}_{i_1 i_2 \ldots i_m} \mid FD^{(S)}_{j_1 j_2 \cdots j_m}\right) = P\left(FS^{(S)}_{i_1 i_2 \ldots i_m}\right) L\left(FD^{(S)}_{j_1 j_2 \cdots j_m} \mid FS^{(S)}_{i_1 i_2 \ldots i_m}\right) \Big/ P\left(FD^{(S)}_{j_1 j_2 \cdots j_m}\right) \tag{5}$$

The posterior probabilities are survival/failure state probabilities conditional on the survival/failure states indicated by the SHM system. The probability of survival/failure state indication itself can be calculated as follows:

$$P\left(FD^{(S)}_{i_1 i_2 \ldots i_m}\right) = \sum_{j_1 j_2 \cdots j_m \in \{0,1\}} P\left(FS^{(S)}_{j_1 j_2 \cdots j_m}\right) L\left(FD^{(S)}_{i_1 i_2 \ldots i_m} \mid FS^{(S)}_{j_1 j_2 \cdots j_m}\right) \tag{6}$$

where in the above summation extends over all possible actual system survival/failure states.

Probabilities $L\left(FD^{(S)}_{i_1 i_2 \ldots i_m} \mid FS^{(S)}_{j_1 j_2 \cdots j_m}\right)$ are the likelihoods of survival/failure state indications by the SHM system given a certain survival/failure state actually occurred. They are specifications of the SHM system performance and must be known, estimated or assumed at the time a decision to adopt an SHM system is being made to be able to quantify the expected benefits. In the context of large and unique structural systems like bridges, there will be in practice limited (perhaps none) experimental validation data to establish these likelihoods. Note we try to make inferences about the performance of an SHM system before we actually deploy it on the structure, thus have no 'hard' measured data at the point of decision making. Even available data or experience from 'similar' structures will have limitations. Circumventing this major challenge will likely require relying on extensive probabilistic numerical simulations where the given structural system with expected uncertainties will be simulated for random combinations of structural properties and load cases to determine its 'virtual' responses from which the SHM system will try to make survival/failure state inferences.

3.3 SHM system topology optimisation problem formulation

The total expected cost or risk can be minimised via choosing an SHM system with an optimal sensor placement topology by solving the following optimisation problem:

$$ST_{(i_1, i_2, \ldots i_n)_{opt}} = \min_{i_1, i_2, \ldots i_n \in \{0,1\}} E_{FD} \min_{k \in \{0,1\}} E_{FS|FD}\left[C\left(ST_{i_1, i_2, \ldots i_n}, FD_{j_1, j_2, \ldots j_m}, AC_k, FS_{l_1, l_2, \ldots l_m}\right)\right] \tag{7}$$

The above equation also indicates how one calculates the expected cost/risk using a decision tree: Working from right to left, at each random event node (circle in Figure 3) the expected value of costs is taken (i.e. cost times its probability of eventuating), and at each decision/action node (square in Figure 3) the minimum of all the incoming branches is taken to minimize the risk. It is called the roll-back technique (Raiffa and Schlaifer 1961).

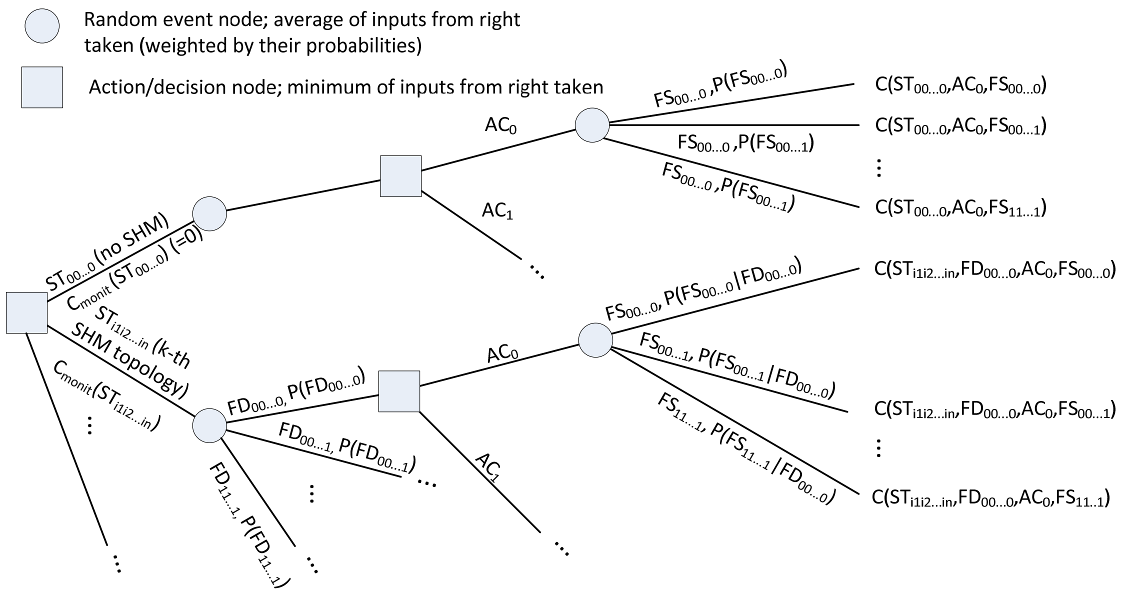

Figure 3. Decision tree for pre-posterior Bayesian analysis for sensing system topology optimisation.

4 ILLUSTRATIVE EXAMPLE

Consider a non-redundant truss structure shown in Figure 4 subjected to a vertical load Q. The top member will be referred to as the tension member and using symbol T, while the bottom member as the compression member and using symbol C, respectively. The material of the truss is perfect elastic-plastic and failure occurs when the yielding point is reached. The mean capacity of the tension member is μ_T=170 kN and standard deviation is σ_T=10 kN, whereas for the compression member μ_C=200 kN and σ_C=10 kN. For the load Q, the mean is denoted by μ_Q (the numerical values of this statistical moment will vary in our example) and standard deviation is σ_Q=10 kN, respectively. All variables are assumed to be independent and normally distributed.

The possible system survival/failure modes are also shown in Figure 4. They can be coded as follows:

$$\mathbf{FS}^T = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} C \\ T \end{matrix} \tag{8}$$

The prior probability of failure of the tension member T, from elementary structural mechanics and reliability theory (Melchers 1999), is:

$$P\left(FS_1^{(T)}\right) = \Phi\left(4/3\,\mu_Q - \mu_T \Big/ \sqrt{\left(4/3\,\sigma_Q\right)^2 + \sigma_T^2}\right) \tag{9}$$

where Φ denotes the standard Gaussian cumulative probability distribution function. A similar

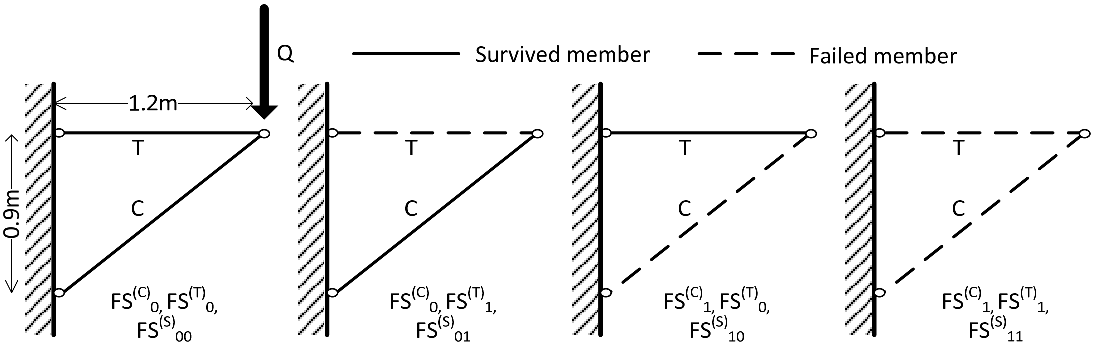

Figure 4. Truss structure and its survival/failure modes.

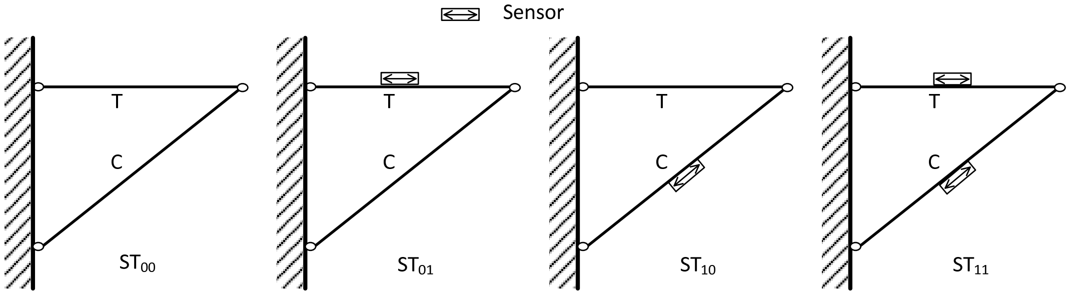

Figure 5. Candidate SHM system topologies.

calculation for the compression member gives $P\left(FS_1^{(C)}\right)$. The probabilities of the various survival/failure states for the system can be worked out based on the survival/failure probabilities of individual members. For example, for the state where both members survive:

$$P\left(FS_{00}^{(S)}\right)=P\left(FS_0^{(C)}\cap FS_0^{(T)}\right)=P\left(FS_0^{(C)}\right)P\left(FS_0^{(T)}\right)=\left(1-P\left(FS_1^{(C)}\right)\right)\left(1-P\left(FS_1^{(T)}\right)\right) \quad (10)$$

taking into account that survival of each member is independent of each other.

Assume now that we have a monitoring system measuring directly member strains or elongations such that yielding of each member on which a sensor is installed can be directly detected. Using the previously introduced notation, the various possible SHM topologies can be coded as follows (see also Figure 5):

$$\mathbf{ST}^T=\begin{bmatrix}0 & 0 & 1 & 1\\ 0 & 1 & 0 & 1\end{bmatrix}\begin{matrix}C\\ T\end{matrix} \quad (11)$$

This also means that an indication of each member state (survived or failed) is independent of the indication for the other member. The performance of each sensor (or more generally a monitoring subsystem) installed on a member can be characterised by the likelihoods of correct and erroneous member survival/failure state indication; for example, for the tension member $L\left(FD_1^{(T)}\,|\,FS_1^{(T)}\right)$ (true positives), $L\left(FD_0^{(T)}\,|\,FS_0^{(T)}\right)$ (true negatives), $L\left(FD_1^{(T)}\,|\,FS_0^{(T)}\right)$ (false positives), and $L\left(FD_0^{(T)}\,|\,FS_1^{(T)}\right)$ (false negatives).

Thus, the overall likelihoods of the various indications for the system, taking into account the assumed independencies of member survival/failure state indications, are as follows:

$$L\left(FD_{ij}^{(S)} \mid FS_{kl}^{(S)}\right) = L\left(FD_{i}^{(T)} \mid FS_{k}^{(T)}\right) L\left(FD_{j}^{(C)} \mid FS_{l}^{(C)}\right) \tag{12}$$

We further assume in this example that the likelihoods of true positive and true negative member survival/failure state indications are $L\left(FD_{11}^{(C)} \mid FS_{11}^{(S)}\right) = L\left(FD_{00}^{(C)} \mid FS_{00}^{(C)}\right) = L\left(FD_{11}^{(T)} \mid FS_{11}^{(T)}\right) = L\left(FD_{00}^{(T)} \mid FS_{00}^{(T)}\right) = 0.999$ (implying a very reliable performance of the SHM system).

The costs or consequences we assume to be as follows:

- When the monitoring system does not indicate failure and there is no actual failure (true negative), and therefore there is no interruption to the normal operations, there are no consequences either.
- When the monitoring system indicates failure but there is no actual failure (false positive), an unnecessary operation interruption occurs, which costs 100 (we use here some arbitrary 'non-dimensional currency').
- When the monitoring system indicates failure and there is actual failure (true positive), there is some damage and operation interruption occurs, but further damage is prevented, and this costs 500.
- When the monitoring system does not indicate failure but there is actual failure (false negative), severe additional cascading damage occurs and as a result operations are interrupted for a long period of time, which costs 10,000.

Table 1 present the results of a study where we vary the mean value of the external load *Q* between 80 and 120 kN (or, equivalently, prior failure probabilities) and observe how these variations change the overall expected cost/risk for different SHM system topologies. Firstly, it can be seen that the prior probabilities of failure are generally higher for the compressive member.

In calculating the total expected costs, we excluded the cost of the SHM system itself, so by subtracting the risk when the various SHM topology choices are used from the risk of the no-SHM scenario (topology [0 0]) we can calculate up to how much one should be willing to spend on a given SHM system to achieve a positive cost-benefit ratio. It can be seen from Table 1 that using any SHM system reduces the risk. As expected, if one had to choose to instrument only one member it should be the more at-risk compressive member (topology [1 0]). Adding an additional sensor (topology [1 1]) will always lead to further risk reduction. We leave open the question whether doing so will incur additional cost that may potentially outweigh the benefits of the lowered risk. Note that this will be more likely for a very safe system (i.e. with a low prior failure probability) – in this case failure is rather unlikely (and consequently prior risk is low) that adoption of an SHM system will not be warranted, because it will very unlikely ever indicate failure. (At the other end of the spectrum, there will be situations where the structural system has a high prior failure probability – in such cases, one should already stop operations based on prior risk assessment. SHM is really worthwhile for the intermediate range of prior failure probabilities. This is yet another example that priors influence strongly pre-posterior analysis outcomes.)

5 CONCLUSIONS

This paper outlines a theory for optimising SHM system topology based on the maximising the value of information that can be extracted from measurements for manging the risk of structural failure. The value of SHM information is in overall reduction of the expected risk and is calculated

Table 1. Prior failure probabilities and risk for varying external load and different SHM system topologies

	Mean of load μ_Q (kN)	80	90	100	110	120
	Prior *T* failure prob. $P\left(FS_1^{(T)}\right)$	$0.0^{8}13$	$0.0^{5}13$	$0.\ 0.0^{3}3$	0.01.4	0.17
	Prior *C* failure prob. $P\left(FS_1^{(C)}\right)$	$0.0^{9}3$	$0.0^{5}18$	$0.0^{2}10$	0.061	0.50
SHM topology	Prior system failure prob. $P\left(FS_{01}^{(S)} \cup FS_{10}^{(S)} \cup FS_{11}^{(S)}\right)$	$0.0^{8}17$	$0.0^{5}32$	$0.0^{2}13$	0.075	0.57
[0 0] (no SHM)	Risk	1.00	1.03	14.00	131.9	336.8
[0 1] (T instrumented)	Risk	$0.0^{8}17$	0.03	10.37	129.9	334.8
[1 0] (C instrumented)	Risk	$0.0^{8}17$	0.03	3.49	129.9	334.8
[1 1] (C&T instrumented)	Risk	$0.0^{8}17$	0.03	0.86	38.3	298.3

using the pre-posterior Bayesian analysis. The paper explains how system survival/failure state probabilities can be obtained from local member- or cross-section-level survival/failure probabilities. The pre-posterior Bayesian analysis is then used to update these survival/failure probabilities using imperfect SHM information. The decision tree analysis is used to minimise the total risk. Finally, the sensing system topology optimisation problem is stated mathematically. A simple analytical example using a truss structure with strain gauge-based SHM system rounds up the paper and illustrates the concepts discussed. Using the example, it is shown how one can analyse systematically how to best allocate limited resources by focusing monitoring activities where the risk and potential for its reduction are the greatest and how this depends on the prior failure risk profile of the structural system.

6 REFERENCES

Kammer, DC, and Tinker, ML, 2004, Optimal placement of triaxial accelerometers for modal vibration tests. *Mechanical Systems and Signal Processing,* 18(1): 29-41.

Melchers, RE, 1999, *Structural Reliability Analysis and Prediction*, *2nd Edition*. New York: John Wiley & Sons, Inc.

Meo, M, and Zumpano, G, 2005, On the optimal sensor placement techniques for a bridge structure. *Engineering Structures,* 27(10): 1488-1497.

Omenzetter, P, 2017, A framework for quantifying and optimizing the value of seismic monitoring of infrastructure. Proceedings of SPIE's Smart Structures and Materials/Nondestructive Evaluation and Health Monitoring 2017, Portland, OR, 101691A-1-9.

Raiffa, H, and Schlaifer, R. 1961. *Applied Statistical Decision Theory*. Harvard University.

Straub, D. (2014). Value of information analysis with structural reliability methods. *Structural Safaty,* 49: 75–85.