An edge cracked frame finite element for analysis of cracked structures and inverse crack detection

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ABSTRACT: A cracked frame element based on the force formulation method and compliance concept is proposed. The formulation is developed according to the Euler-Bernoulli theory. There is an open stable edge crack in the element, which its effects on the local flexibility is computed by utilizing linear elastic fracture mechanics principles. This additional flexibility is added to the non-cracked element flexibility matrix which is derived by force-formulation method in a basic coordinate system without rigid body motions. The suggested element is applicable for non-prismatic frames and the effects of the internal forces interactions are also included. A consistent mass matrix is also derived for the element.

The proposed element is applicable for static and dynamic analysis of cracked and non-cracked frame structures. In addition, due to its capability in calculating free vibration responses accurately, the presented element can be used for inverse crack detection techniques.

1 *INTRODUCTION*

Cracking is one of the most common damages that occurs during service life of engineering structures and in some cases, causes considerable reduction in their load bearing capacity and even might leads to collapse. Thus, crack detection and analysis of cracked structures are important and interesting topics for researchers (Dimarogonas, 1996). Because of complexity of crack behavior, different numerical approaches are used to analyze cracked structures. Among these approaches, finite element method provides better results according to its capabilities. As a results, many finite element studies are conducted on the behavior of the cracked frames. In the first attempts to formulate a cracked frame element, Tharp (1987) proposed an edge cracked beam-column element. Saavedra and Cuitino (2002) computed a new stiffness matrix for cracked elements by calculation of the additional flexibility due to cracking. In another research, Viola et al. (2001) suggested a displacement-based element for analysis of cracked Timoshenko beams. They established their element based on the local flexibility concept introduced by Okamura (1969) and also present a consistent mass matrix for the element. In 2007, Skrinar and Plibersek formulated a finite element for slender cracked beam under transverse loading. In another research, Skrinar (2009) suggested a finite element for stability analysis of cracked frame elements. Skrinar and Lutar (2012) proposed a three node beam element for analysis of cracked slender beam located on Winkler's foundation.

It is evident that cracking reduce local stiffness of structures. From the force standpoint, it is equivalent to increase in local flexibility. Based on this principle, it is possible to calculate additional flexibility as a function of the strain energy release rate. In this method of formulation, crack is not considered as a unique phenomenon and only its effect on the flexibility of the structure is computed. Due to utilizing linear fracture mechanics relations in this type of formulation, the resulted elements are only valid for the structures made of brittle or semi-brittle materials in which nonlinear zone around the crack tip is small enough to be neglected. Besides this limitations, this formulation provide some advantages. For example, only one element is needed to model crack in the structure. This leads to considerable decrease in the computation time and cost.

In this paper, a new force-based finite element for the analysis of cracked non-prismatic frame elements will be presented based on the Euler-Bernoulli beam theory. For this purpose, first the non-cracked element will be formulated in a basic coordinate system without rigid body motions. Flexibility matrix of the element will be calculated by means of exact force interpolation matrix. Then, the additional flexibility matrix due to crack will be derived as a function of strain energy release rate using linear fracture mechanics relations. The strain energy release rate, itself is a function of the crack tip stress intensity factors. Flexibility matrix of the cracked element is sum of the non-cracked and additional flexibilities. Then, a flexibility consistent mass matrix will be formulated for the element.

2 *NON-CRACKED ELEMENT FORMULATION*

A 2D beam finite element in global and local coordinate systems possesses six degree of freedom, which are depicted in figure (1).

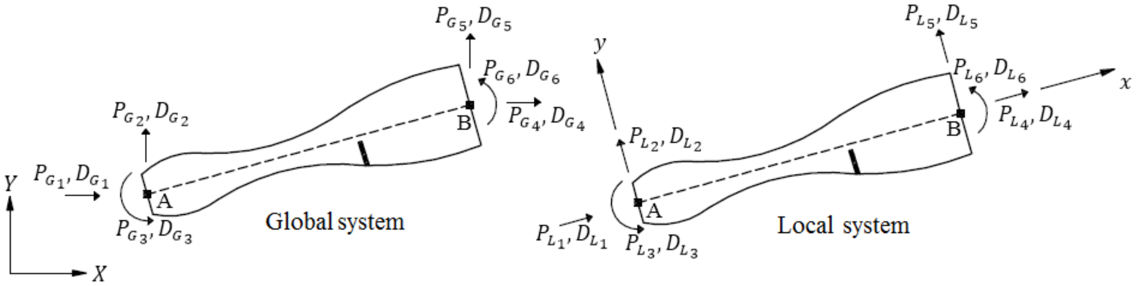

Figure 1. The cracked beam element in the global and local coordinates.

Global nodal forces and displacements are as follows:

$$P_G = \{P_{G_1} \quad \cdots \quad P_{G_6}\}^T \tag{1}$$

$$D_G = \{D_{G_1} \quad \cdots \quad D_{G_6}\}^T \tag{2}$$

The local nodal forces and displacements are also named P_L and D_L, respectively:

$$P_L = \{P_{L_1} \quad \cdots \quad P_{L_6}\}^T \tag{3}$$

$$D_L = \{D_{L_1} \quad \cdots \quad D_{L_6}\}^T \tag{4}$$

Because of the presence of rigid body modes in the global and local coordinates, there is no flexibility matrix for the element in these systems. To remove this problem the basic coordinate system demonstrated in figure (2) will be used for non-cracked element formulation. The basic nodal forces and displacements in this system are as follows:

$$P_B = \{P_{B_1} \quad P_{B_2} \quad P_{B_3}\}^T = \{N_B \quad M_A \quad M_B\}^T \tag{5}$$

$$D_B = \{D_{B_1} \quad D_{B_2} \quad D_{B_3}\}^T = \{u_B \quad \theta_A \quad \theta_B\}^T \tag{6}$$

For an Euler-Bernoulli beam element, the displacement vector of neutral axis can be defined in the following form:

$$U(x) = \begin{Bmatrix} u(x) \\ v(x) \end{Bmatrix} \tag{7}$$

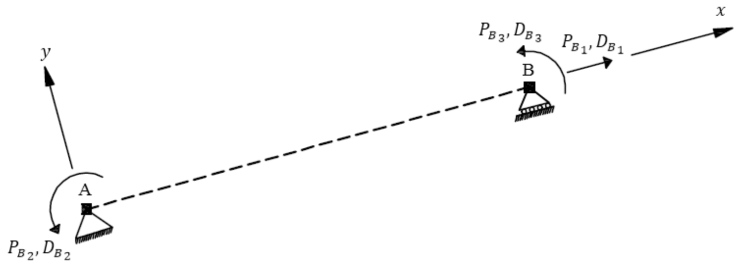

Figure 2. The suggested element in the basic system without rigid body motions.

Where, u and v are axial and lateral displacements, respectively. Based on the Euler-Bernoulli theory, deformation or generalized strains are as comes:

$$d(x) = \begin{Bmatrix} \varepsilon(x) \\ \kappa(x) \end{Bmatrix} = \begin{Bmatrix} \frac{du(x)}{dx} \\ \frac{d^2v(x)}{dx^2} \end{Bmatrix} \tag{8}$$

Solving the differential equations which governs behavior if the beam by taking advantage of the boundary conditions according to the basic coordinate system results in the following relation between internal forces and the basic nodal forces:

$$S(x) = \begin{Bmatrix} N(x) \\ M(x) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \xi - 1 & \xi \end{bmatrix} . \begin{Bmatrix} P_{B_1} \\ P_{B_2} \\ P_{B_3} \end{Bmatrix} = b(x).P_B \quad , \xi = \frac{x}{L} \tag{9}$$

Where L is the element length, *S(x)* indicates the vector of internal forces which is related to the basic nodal forces by means of the force interpolation matrix, *b(x)*. It must be noted that as opposed to the assumed displacement interpolation functions in the displacement-based formulation, this force interpolation matrix is exact and independent of cross section variations. Finally, the flexibility matrix of the element without crack in the basic system are derived from the following equation:

$$[f_B{}^{nc}] = \int_0^L b(x)^T . [k_s]^{-1} . b(x) dL \tag{10}$$

This flexibility matrix regardless of the numerical integration error is exact, because the exact force interpolation matrix are used in formulation. In this relation, k_s is the section stiffness matrix, which for a non-prismatic beam with general cross section is derived from the coming relation:

$$[k_s] = \int_A \begin{bmatrix} E & -yE \\ -yE & y^2E \end{bmatrix} dA \tag{11}$$

In equation (11), E is young modulus of the beam material.

3 *ADDITIONAL FLEXIBILITY DUE TO CRACK*

Cracking reduce local stiffness of structures. It is equivalent to increase in the flexibility and can be computed as a function of released strain energy due to cracking. Thus, first the released energy, U_r, must be computed using the following relation:

$$U_r = \frac{1}{E}\int_{A_c}\left({K_I}^2 + {K_{II}}^2\right)dA_c \tag{12}$$

In this equation, K_I and K_{II} are the crack tip stress intensity factors corresponding to the first and second fracture modes, respectively. A_c is the area of crack faces. The first fracture mode at the crack tip occurs due to axial force and bending moment. For the axial force, N, the first stress intensity factor of an edge crack is derived using the next equation (Tada et al. , 2000):

$$K_{I_N} = \frac{N}{A}\sqrt{\pi . a_c} . h(\frac{a_c}{b}) \tag{13}$$

In the previous relation, a_c is the crack length and A is cross section area of the beam. b is the beam width in the crack location and $h(\frac{a_c}{b})$ is an empirical or analytical modification function that take into account the effects of finite dimensions of the structure. In this study, the formula proposed by Tada et al. (2000) will be used:

$$h\left(\frac{a_c}{b}\right) = 0.265\left(1 - \frac{a_c}{b}\right)^4 + \frac{0.857 + 0.265\frac{a_c}{b}}{\left(1 - \frac{a_c}{b}\right)^{\frac{3}{2}}} \tag{14}$$

The opening mode stress intensity factor caused by bending moment, M, computed as follows (Tada et al., 2000):

$$K_{I_M} = \frac{6M}{tb}\sqrt{\pi . a_c} . g(\frac{a_c}{b}) \tag{15}$$

which t is the beam thickness and $g(\frac{a_c}{b})$ is the modification function and defined as comes in the next relation:

$$g\left(\frac{a_c}{b}\right) = 1.122 - 1.4\left(\frac{a_c}{b}\right) + 7.33\left(\frac{a_c}{b}\right)^2 - 13.08\left(\frac{a_c}{b}\right)^3 + 14\left(\frac{a_c}{b}\right)^4 \tag{16}$$

The sliding mode stress intensity factor is taken equal to zero, because in the Euler-Bernoulli beam theory the shear deformations are neglected. Replacing equation (13) to (16) in equation (12) results in the next relation:

$$U_C = \frac{1}{E}\int_{A_c}\left(K_{I_N} + K_{I_M}\right)^2 dA_c = \frac{1}{E}\int_{A_c}\left(\frac{N}{A}\sqrt{\pi . a_c} . h(\frac{a_c}{b}) + \frac{6M}{tb}\sqrt{\pi . a_c} . g(\frac{a_c}{b})\right)^2 dA_c \tag{17}$$

This equation can be expanded furthur by replacing internal forces in terms of the basic nodel forces from equation (9):

$$U_C = \frac{1}{E}\int_0^{a_0}\left(\left(\frac{P_{B_1}}{A}\sqrt{\pi . a_c} . h(\frac{a_c}{b}) + \frac{6\left((\xi_c - 1).P_{B_2} + \xi_c . P_{B_3}\right)}{tb}\sqrt{\pi . a_c} . g(\frac{a_c}{b})\right)^2 + \right) t da_c \tag{18}$$

where a_0 and ξ_c are crack length and natural coordinate of the crack location in basic system, respectively.Finally, the additonal flexibility matrix is derived by differentiating the released strain energy with respect to the basic forces:

$$\left[{f_B}^C\right] = \frac{\partial^2 U_C}{\partial P_{B_i} . \partial P_{B_j}} = \begin{bmatrix} {f_{11}}^C & {f_{12}}^C & {f_{13}}^C \\ {f_{21}}^C & {f_{22}}^C & {f_{23}}^C \\ {f_{31}}^C & {f_{32}}^C & {f_{33}}^C \end{bmatrix} \quad for\ i,j = 1,2,3 \tag{18}$$

$$\begin{cases} {f_{11}}^{c} = \frac{\partial^2 U_C}{\partial P_{B_1}.\partial P_{B_1}} = \frac{2t}{E}\int_0^{a_0}\left[\frac{(\pi.a_c)}{A^2}.h^2\left(\frac{a_c}{b}\right)\right]da_c \\ {f_{12}}^{c} = {f_{21}}^{c} = \frac{\partial^2 U_C}{\partial P_{B_1}.\partial P_{B_2}} = \frac{2t}{E}\int_0^{a_0}\left(\frac{6(\xi_c-1)}{Atb}.h\left(\frac{a_c}{b}\right).g\left(\frac{a_c}{b}\right)\right)(\pi.a_c)da_c \\ {f_{13}}^{c} = {f_{31}}^{c} = \frac{\partial^2 U_C}{\partial P_{B_1}.\partial P_{B_3}} = \frac{2t}{E}\int_0^{a_0}\left(\frac{6\xi_c}{Atb}.h\left(\frac{a_c}{b}\right).g\left(\frac{a_c}{b}\right)\right)(\pi.a_c)da_c \\ {f_{22}}^{c} = \frac{\partial^2 U_C}{\partial P_{B_2}.\partial P_{B_2}} = \frac{2t}{E}\int_0^{a_0}\left(\frac{36(\xi_c-1)^2}{t^2b^2}.g^2\left(\frac{a_c}{b}\right)\right)(\pi.a_c)da_c \\ {f_{23}}^{c} = {f_{32}}^{c} = \frac{\partial^2 U_C}{\partial P_{B_1}.\partial P_{B_2}} = \frac{2t}{E}\int_0^{a_0}\left(\frac{36(\xi_c-1)\xi_c}{t^2b^2}.g^2\left(\frac{a_c}{b}\right)\right)(\pi.a_c)da_c \\ {f_{33}}^{c} = \frac{\partial^2 U_C}{\partial P_{B_3}.\partial P_{B_3}} = \frac{2t}{E}\int_0^{a_0}\left(\frac{36{\xi_c}^2}{t^2b^2}.g^2\left(\frac{a_c}{b}\right)\right)(\pi.a_c)da_c \end{cases} \quad (19)$$

Now the elementflexibility matrix in basic coordinate is derived by adding the additional flexibility to the non-cracked one:

$$[f_B] = [{f_B}^{nc}] + [{f_B}^{c}] \quad (20)$$

4 TRANSFORMATION OF THE STIFFNESS MATRIX TO GLOBAL COORDINATES

Because of inherent difficulties of the force methods, complete analysis of a structure by force method is very complicated. Therefore, the flexibility matrix is inverted to be used in a stiffness analysis framework:

$$[k_B] = [f_B]^{-1} \quad (21)$$

The following simple transformation relation provides stiffness matrix in the local system, , k_L:

$$[k_L] = [T]^T.[k_B].[T] \quad (22)$$

where T is the basic to local coordinates transformation matrix and is defined as follows:

$$[T] = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1/L & 1 & 0 & -1/L & 0 \\ 0 & 1/L & 0 & 0 & -1/L & 1 \end{bmatrix} \quad (23)$$

Finally, the global and the local stiffness matrices are related together by the familiar transformation matrix, R:

$$[k_G] = [R]^T.[k_L].[R] \quad (24)$$

$$[R] = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (25)$$

5 CONSISTENT MASS MATRIX

Based on the Virtual displacement principal, works performed by internal and external inertia forces are equal. Therefore, the following relation in basic coordinate system can be established:

$$\int_0^L \{ {P_I}_i^{\,T}(x).\,\delta U(x)\}\, dx = {P_I}_e^{\,T}.\,\delta D_B \tag{26}$$

in which $\delta U(x)$ is the virtual internal displacements. ${P_I}_i$ and ${P_I}_e$ are internal and external inertia forces which are defined as follows:

$${P_I}_i(x) = m_s(x).\,\delta \ddot{U}(x) \tag{27}$$

$${P_I}_e = m.\,\delta \ddot{D}_B \tag{28}$$

In these equations, $\delta\ddot{D}$ and $\delta\ddot{U}$ are nodal and internal accelerations. Is section mass matrix and is defined as follows:

$$m_s(x) = \begin{bmatrix} \int_A \rho\, dA & -\int_A \rho.y\, dA \\ -\int_A \rho.y\, dA & \int_A \rho.y^2\, dA \end{bmatrix} \tag{29}$$

Using equations (27) to (29), the equation (26) is expanded in the succeeding form:

$$\int_0^L \{ \delta\ddot{U}^T(x).\,m_s(x).\,\delta U(x)\}\, dx = \delta\ddot{D}_B^{\,T}.\,m.\,\delta D_B \tag{30}$$

In order to relate internal and nodal displacements together, again the virtual work principle will be used. A set of virtual forces are assumed to be applied to the beam at an arbitrary location, x_V. Therefore, the following relation is derived:

$$\delta {P_B}^T.\,D_B + \delta {P_V}^T.\,U(x_V) = \int_0^L \delta S^T(x, x_V).\,d(x)\, dx \tag{31}$$

In this equality, δP_V is vector of virtual forces applied at point x_V to the beam. δP_B and δS are the nodal and internal forces, respectively, that are in equilibrium with the virtual forces. To derive the relation between internal and nodal displacements, it is necessary to compute δP_B and δS as a function of applied virtual forces. To do so, a system of forces must be considered. Because the ultimate goal of the formulation is to formulate mass matrix of a general non-prismatic beam element, any force system that only satisfies equilibrium conditions, would be an admissible virtual system. Therefore a system is selected such that the following relation exists between the internal forces and the applied virtual forces:

$$\delta S(x, x_V) = b_V(x, x_V).\,\delta P_V \tag{32}$$

$$b_V(x,\xi) = \begin{cases} \frac{1}{2}.\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} & x \le x_V \\ \frac{1}{2}.\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & x_V \le x \end{cases} \tag{33}$$

The corresponding nodal forces in this system are connected to the virtual forces as follows:

$$\delta P_B = b_B(x, x_V).\,\delta P_V = -\frac{1}{2}.\begin{bmatrix} 1 & 0 & 0 \\ 0 & x_V & 1 \\ 0 & x_V - L & 1 \end{bmatrix}.\begin{Bmatrix} \delta {P_V}_1 \\ \delta {P_V}_2 \\ \delta {P_V}_3 \end{Bmatrix} \tag{34}$$

By replaceing equations (34) and (32) in (31) and some mathemathical efforts, the following relation between internal and nodal dislacements is derived:

$$U_0(x_V) = \left\{ \int_0^L \left[{b_V}^T(x, x_V).\,f_s(x).\,b(x).\,[f_B]^{-1} \right] dx - {b_B}^T(x, x_V) \right\}.\,D_B = N_B(x_V).\,D_B \tag{35}$$

Using the previous relation, equation (30) can be rewritten as follows:

$$\int_0^L \{ \delta \ddot{D}^T . {N_B}^T(x) . m_s(x) . N_B(x) . \delta D_B \} \, dx = \delta {\ddot{D}_B}^T . m . \delta D_B \tag{36}$$

And finally, the mass matrix of the element in the basic coordinates is derived:

$$m = \int_0^L \{ {N_B}^T(x) . m_s(x) . N_B(x) \} \, dx \tag{37}$$

To make the suggested element applicable for dynamic analysis, it is necessary to derive element mass matrix in the global coordinate system. For this purpose, rigid body rotations of the element assumed to be small. Therefore, the angle between element chord in the basic system and x axis of the local coordinates is neglected. According the mentioned assumption, next relation exist between displacements in the two coordinate system:

$$\bar{U}(x) = U(x) + N_{rig}(x) . D_L \tag{37}$$

Where and U are $\bar{U}$ the neutral axis displacement vectors in the basic and the local systems. D_L is the local nodal displacement vector and N_{rig} is an interpolation matrix which is defined as follows:

$$N_{rig}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 - x/L & 0 & 0 & x/L & 0 \\ 0 & -1/L & 0 & 0 & 1/L & 0 \end{bmatrix} \tag{38}$$

The nodal displacements of the two system are connected together by means of transformation matrix, T, which was presented previously by equation (23).

$$D_B = T . D_L \tag{39}$$

Using equations (39) and (35), equation (37) can be expanded in the following form:

$$\bar{U}(x) = N_B(x) . D_B + N_{rig}(x) . D_L = \left(N_B(x) . T + N_{rig}(x) \right) . D_L = N_L(x) . D_L \tag{40}$$

Like the basic system, using virtual work priciple leads the next re;ation for computation of the local mass matrix:

$$m_L = \int_0^L \left\{ \left(N_B(x) . T + N_{rig}(x) \right)^T . m_s(x) . \left(N_B(x) . T + N_{rig}(x) \right) \right\} dx \tag{41}$$

Finally, a simple transformation provides the global element mass matrix:

$$m_G = R^T . m_L . R \tag{42}$$

6 *A NUMERICAL EXAMPLE*

To investigates ability of the proposed element in dynamic analysis of cracked structures, the natural frequencies of a simple cracked beam, which is depicted in figure (3), is computed.

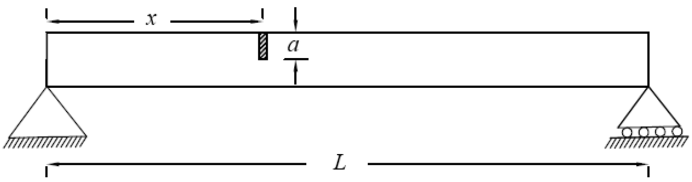

Figure 3. Simple beam with an edge crack.

The length of the beam is 30 cm and it has 2×2 cm rectangular cross section. Its material modulus of elasticity and Poisson's ratio are equal 2060 GPa and 0.3, respectively. The mass density is 7750 kg/m^3. This problem, which was previously analyzed by Kam and Lee (1994). The problem is solved using the proposed element and natural frequencies of the first two modes of vibration are calculated. Figure (4) demonstrates cracked to non-cracked frequencies for the crack at the middle of the beam for various crack lengths. It is evident that the proposed element provide acceptable responses. Also, it can be seen that the error in responses increases by increasing the crack length.

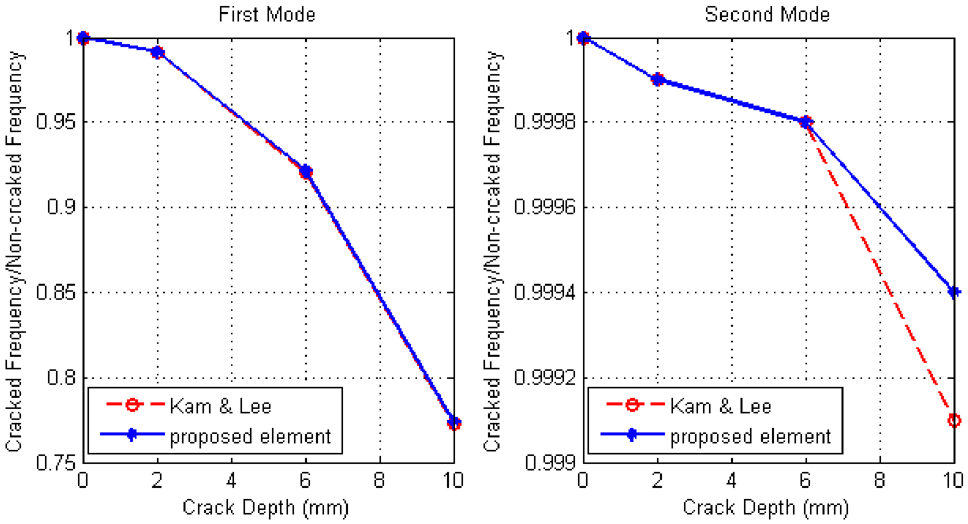

Figure 4. Non-dimensional frequencies vs crack length.

7 CONCLUSION

A two dimensional non-prismatic cracked beam element based on the Euler-Bernoulli theory is presented for static and dynamic analysis of frame structures. The element is formulated based on the force method in a basic coordinate system. The crack effect on the flexibility of the element is calculated as a function of the released strain energy release rate by using linear fracture mechanics principles. Because of using the exact force interpolation matrix in the formulation, the proposed element is exact neglecting the numerical integration error. The element mass matrix is computed also in the basic coordinate system and by means of the force method. A numerical example demonstrate acceptable accuracy of the proposed element for for free vibration analysis of cracked structure. So the proposed element is applicable for vibration based crack detection analysis.

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